647510 **MSC1MatC103x** **Seat No :________**

M.Sc.(Maths.) Semester - 1 (*CBCS*) Examination
Nov/Dec-2023
Topology 1(Core (New))

Time: 2:30 Hours **Marks: 70**

Instructions:

1. Figures to the right indicate marks.
2. All questions are compulsory.

Que.1(A) Answer any Two out of Three. (10)

1) Let $\{\mathcal{T}_\alpha \mid \alpha \in \Lambda\}$ be a family of topologies on a set X. Then show that $\bigcap_{\alpha\in\Lambda}\mathcal{T}_\alpha$ is a topology on X.
2) Let X be a set and $\mathcal{B}$ a basis for a topology $\mathcal{T}$ on X. Then show that $\mathcal{T}$ is the collection of all unions of elements of $\mathcal{B}$.
3) Show that the topologies of $\mathbb{R}_\ell$ and $\mathbb{R}_K$ are both strictly finer than the standard topology on $\mathbb{R}$.

Que.1(B) Answer any One out of Two. (04)

1) Let X be a non-empty set. Let $\mathcal{T}_f$ be the collection of all subsets U of X such that either $X \setminus U$ is finite or whole of X. Then show that $\mathcal{T}_f$ is a topology on X.
2) Let X be a set and $\mathcal{S}$ be a subbasis for a topology on X. Let
$$\mathcal{B} = \{B \subset X \mid B \text{ is intersection of finitely many elements of } \mathcal{S}\}.$$
Then show that $\mathcal{B}$ is a basis for a topology on X.

Que.2(A) Answer any Two out of Three. (10)

1) Let $(X, \mathcal{T})$ be a topological space and $Y \subset X$. Show that the collection
$$\mathcal{T}_Y = \{U \cap Y \mid U \in Y\}$$
is a topology on Y.
2) Let X and Y be topological spaces and $f: X \to Y$ be a function. Then show that the following are equivalent.
   I. f is continuous.
   II. For each $x \in X$ and each neighborhood V of $f(x)$, there is a neighborhood U of x such that $f(U) \subset V$.
3) State and prove the Pasting lemma.

Que.2(B) Answer any One out of Two. (04)

1) Let X and Y be topological spaces. Let
$$\mathcal{B} = \{U \times V \in X \times Y \mid U \text{ is open in } X, V \text{ is open in } Y\}.$$
Then show that $\mathcal{B}$ is a basis for a topology on $X \times Y$.
2) Let X and Y be topological spaces. Show that the collection
$$\mathcal{S} = \{\pi_1^{-1}(U) \mid U \text{ is open in } X\} \cup \{\pi_2^{-1}(V) \mid V \text{ is open in } Y\}$$
is a subbasis for the product topology on $X \times Y$.

Que.3(A) Answer any Two out of Three. (10)

1) Let X be a topological space, let U is open in X and A is closed in X. Then show that $U \setminus A$ is open in X and $A \setminus U$ is closed in X.
2) Let X be a topological space and $A, B \subset X$. Then show that $\overline{A \cup B} = \bar{A} \cup \bar{B}$.
3) Let X and Y be topological spaces, and $A \subset X$ and $B \subset Y$. In the space $X \times Y$ with product topology, show that $\overline{A \times B} = \bar{A} \times \bar{B}$.

Que.3(B) Answer any One out of Two. (04)

1) Let X be a topological space, Y be a subspace of X and $A \subset Y$. Then show that the set A is closed in Y if and only if it is the intersection of a closed set of X with Y.
2) Let X be a topological space and $A \subset X$. Then show that $\bar{A} = A \cup A'$.

Que.4(A) Answer any Two out of Three. (10)

1) Show that the topologies on $\mathbb{R}^n$ induced by the Euclidean metric d and the square metric ρ are the same as the product topology on $\mathbb{R}^n$.
2) Let X be a metrizable space and $A \subset X$. Then show that $x \in \bar{A}$ if and only if there is a sequence of points of A converging to x.
3) Let X and Y be metrizable topological spaces with metrics d_X and d_Y respectively and $f: X \to Y$ be a function. Then show that the continuity of f is equivalent to the condition that given $\varepsilon > 0$ and $x \in X$, there exists $\delta > 0$ such that $d_X(x, y) < \delta \Rightarrow d_Y\big(f(x), f(y)\big) < \varepsilon$.

Que.4(B) Answer any One out of Two. (04)

1) Let (X, d) be a metric space. Then show that the collection
$$\mathcal{B}_d = \{B_d(x, \varepsilon) \mid x \in X, \varepsilon > 0\}$$
is a basis for a topology on X.
2) Let X be a metric space with metric d. Define $\bar{d}: X \times X \to \mathbb{R}$ by
$$\bar{d}(x, y) = \min\{d(x, y), 1\}.$$
Then show that $\bar{d}$ is a metric that induces the same topology on X as d.

Que.5(A) Answer any Two out of Three. (10)

1) Let X be a topological space, let C and D form a separation of X and Y is connected subspace of X. Then show that $Y \subset C$ or $Y \subset D$.
2) Show that the union of collection of connected subspaces of X that have a point in common is connected.
3) Show that the image of a connected space under a continuous map is connected.

Que.5(B) Answer any One out of Two. (04)

1) Show that every path connected space is connected.
2) Let X be a topological space. Then show that every component of X is closed in X.

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